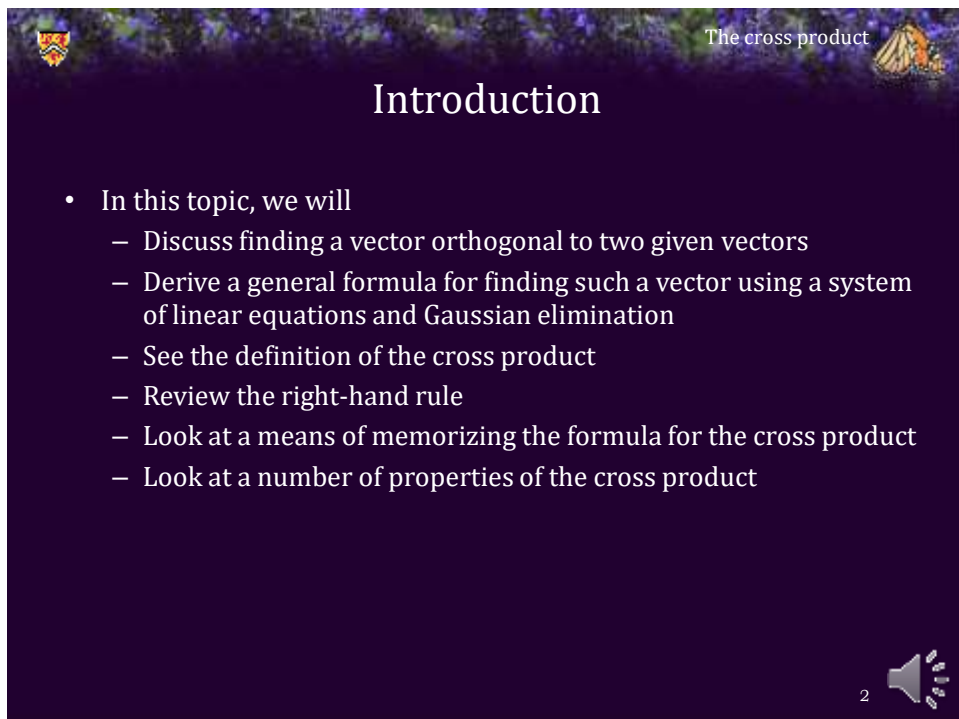




1



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The cross product

Finding an orthogonal vector

- Given two vectors $\mathbf{u}$ and $\mathbf{v}$,
if $\mathbf{u}$ and $\mathbf{v}$ are not parallel, they define a plane
 - That plane has a normal vector
- We want to find this normal vector
 - That vector will be perpendicular to both $\mathbf{u}$ and $\mathbf{v}$
- Suppose this perpendicular vector is $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
 - Thus, both these calculations must be zero:

$$\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle = 0$$

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The cross product

Finding an orthogonal vector

- This gives us two equations in three unknowns w_1, w_2 and w_3
 - The equations are

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0$$

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0$$
 - We immediately note that with two homogenous linear equations, there are infinitely many solutions
- We continue by adding $-v_1/u_1$ times Row 1 onto Row 2:

$$\begin{pmatrix} u_1 & u_2 & u_3 & | & 0 \\ v_1 & v_2 & v_3 & | & 0 \end{pmatrix} \sim \begin{pmatrix} u_1 & u_2 & u_3 & | & 0 \\ 0 & v_2 - \frac{v_1 u_2}{u_1} & v_3 - \frac{v_1 u_3}{u_1} & | & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} u_1 & u_2 & u_3 & | & 0 \\ 0 & u_1 v_2 - v_1 u_2 & u_1 v_3 - v_1 u_3 & | & 0 \end{pmatrix}$$

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The cross product

Finding an orthogonal vector

- How can we find the solutions for this?

$$\begin{pmatrix} u_1 & u_2 & u_3 & | & 0 \\ 0 & u_1v_2 - v_1u_2 & u_1v_3 - v_1u_3 & | & 0 \end{pmatrix}$$
- One solution to $3y + 7z = 0$ is to let $y = -7\alpha$ and $z = 3\alpha$
- Lets do this above:

$$w_2 = -\alpha(u_1v_3 - v_1u_3) = \alpha u_3v_1 - \alpha u_1v_3$$

$$w_3 = \alpha(u_1v_2 - v_1u_2) = \alpha u_1v_2 - \alpha u_2v_1$$
- Substituting these into Equation 1, we get

$$u_1w_1 + u_2(\alpha u_3v_1 - \alpha u_1v_3) + u_3(\alpha u_1v_2 - \alpha u_2v_1) = 0$$

$$u_1w_1 + \alpha u_2u_3v_1 - \alpha u_1u_2v_3 + \alpha u_1u_3v_2 - \alpha u_2u_3v_1 = 0$$

$$u_1w_1 - \alpha u_1u_2v_3 + \alpha u_1u_3v_2 = 0$$

$$w_1 = \alpha u_2v_3 - \alpha u_3v_2$$

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The cross product

The cross product

- Thus, we have our solutions:

$$\mathbf{w} = \alpha \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$
- We could choose any α we want, but let us choose $\alpha = 1$
- We will define this solution as the *cross product* of $\mathbf{u}$ and $\mathbf{v}$:

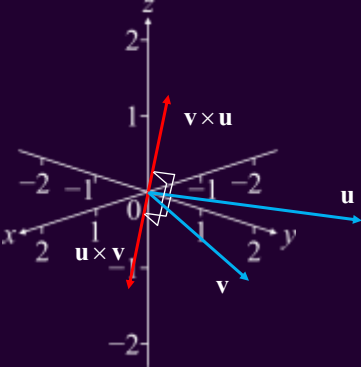
$$\mathbf{u} \times \mathbf{v} \stackrel{\text{def}}{=} \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$
- Noticed if we used $\alpha = -1$, the vector would still be perpendicular, but pointing in the other direction

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The cross product

- Graphically, if we have two vectors $\mathbf{u}$ and $\mathbf{v}$
 - The cross product $\mathbf{u} \times \mathbf{v}$ will be orthogonal to both
 - The cross product $\mathbf{v} \times \mathbf{u}$ will also be orthogonal to both, but point in the opposite direction



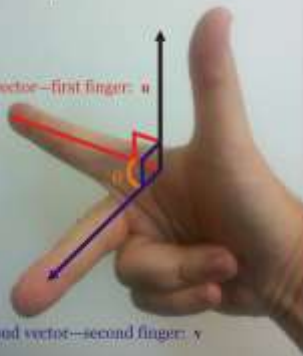
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The right-hand rule

- We have therefore chosen one of two orientations with
 - We will say that this satisfies the *right-hand rule* according to these rules:

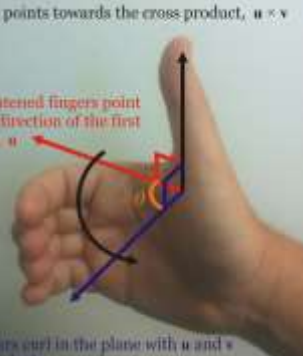
Cross product direction—thumb: $\mathbf{u} \times \mathbf{v}$



First vector—first finger: $\mathbf{u}$

Second vector—second finger: $\mathbf{v}$

Thumb points towards the cross product, $\mathbf{u} \times \mathbf{v}$



Straightened fingers point in the direction of the first vector, $\mathbf{u}$

Fingers curl in the plane with $\mathbf{u}$ and $\mathbf{v}$ towards the second vector, $\mathbf{v}$

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The cross product

An aide-mémoire

- This is an ugly formula....

$$\mathbf{u} \times \mathbf{v} \stackrel{\text{def}}{=} \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

- This is an easier *aide-mémoire* to memorize this formula

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The cross product

Example

- What is the cross product of these two vectors?

$$\mathbf{u} = \begin{pmatrix} -2.1 \\ 0.1 \\ 1.5 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 0.8 \\ -0.3 \\ 0.7 \end{pmatrix} \quad \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 0.52 \\ 2.67 \\ 0.55 \end{pmatrix}$$

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The cross product

Anti-symmetric

- This is not multiplication, for with multiplication $\alpha\beta = \beta\alpha$

Theorem

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

Proof:

$$\mathbf{u} \times \mathbf{v} \stackrel{\text{def}}{=} \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \quad \mathbf{v} \times \mathbf{u} \stackrel{\text{def}}{=} \begin{pmatrix} v_2 u_3 - v_3 u_2 \\ v_3 u_1 - v_1 u_3 \\ v_1 u_2 - v_2 u_1 \end{pmatrix} = \begin{pmatrix} u_3 v_2 - u_2 v_3 \\ u_1 v_3 - u_3 v_1 \\ u_2 v_1 - u_1 v_2 \end{pmatrix}$$

$$= - \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = -(\mathbf{u} \times \mathbf{v}) \quad \mathcal{QED}$$

- This property is described as being *anti-symmetric*
- Recall the real inner product is symmetric: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

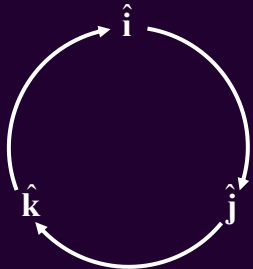
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The cross product

Canonical basis vectors

- The cross products of the canonical basis vectors are as follows:

$$\begin{aligned} \hat{\mathbf{i}} \times \hat{\mathbf{j}} &= \hat{\mathbf{k}} & \hat{\mathbf{j}} \times \hat{\mathbf{k}} &= \hat{\mathbf{i}} & \hat{\mathbf{k}} \times \hat{\mathbf{i}} &= \hat{\mathbf{j}} \\ \hat{\mathbf{j}} \times \hat{\mathbf{i}} &= -\hat{\mathbf{k}} & \hat{\mathbf{k}} \times \hat{\mathbf{j}} &= -\hat{\mathbf{i}} & \hat{\mathbf{i}} \times \hat{\mathbf{k}} &= -\hat{\mathbf{j}} \end{aligned}$$



	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$
$\hat{\mathbf{i}}$	1	0	0	0	1	0
$\hat{\mathbf{j}}$	0	1	0	0	0	1
$\hat{\mathbf{k}}$	0	0	1	0	0	0

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The cross product

Canonical basis vectors

- Any cross product with the zero vector is the zero vector:

$$\mathbf{0} \times \mathbf{v} = \mathbf{0}$$

$$\mathbf{u} \times \mathbf{0} = \mathbf{0}$$

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The cross product

Bilinear

- Recall for the real inner product, it is bilinear:

$$\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$$

$$\langle \mathbf{u}, \beta \mathbf{v} \rangle = \beta \langle \mathbf{u}, \mathbf{v} \rangle$$

Theorem

The cross product is bilinear:

$$(\alpha \mathbf{u}) \times \mathbf{v} = \alpha (\mathbf{u} \times \mathbf{v})$$

$$\mathbf{u} \times (\beta \mathbf{v}) = \beta (\mathbf{u} \times \mathbf{v})$$

– The proof is left to the viewer...

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The cross product

Magnitude

- Recall for the real inner product,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \cos(\theta)$$
 where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

Theorem

$$\|\mathbf{u} \times \mathbf{v}\|_2 = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \sin(\theta)$$

Proof:

The proof is in the course text,
but it is nothing more than an exercise in algebra...

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The cross product

Non-associative

- Recall that field addition, field multiplication and vector addition are all associative:

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \quad \alpha(\beta\gamma) = (\alpha\beta)\gamma \quad \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$
- Recall also that field subtraction, field division and vector subtraction are all not associative:

$$\alpha - (\beta - \gamma) \neq (\alpha - \beta) - \gamma \quad \alpha \div (\beta \div \gamma) \neq (\alpha \div \beta) \div \gamma \quad \mathbf{u} - (\mathbf{v} - \mathbf{w}) \neq (\mathbf{u} - \mathbf{v}) - \mathbf{w}$$

Theorem

The cross product is not associative.

Proof:

$$\hat{\mathbf{i}} \times (\hat{\mathbf{i}} \times \hat{\mathbf{j}}) = \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

$$(\hat{\mathbf{i}} \times \hat{\mathbf{i}}) \times \hat{\mathbf{j}} = \mathbf{0} \times \hat{\mathbf{j}} = \mathbf{0} \quad \mathcal{QED}$$

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The cross product

Observations

- As the angle between $\mathbf{u}$ and $\mathbf{v}$ decreases, the magnitude of the cross product decreases
- The cross product $\mathbf{u} \times \mathbf{v}$ is the zero vector if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other
- If $\mathbf{u}$ and $\mathbf{v}$ are already orthonormal, then $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ form an orthonormal basis for $\mathbf{R}^3$



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The cross product

Summary

- Following this topic, you now
 - Understand the purpose of the cross product
 - You understand how to find the cross product
 - Have seen an *aide-mémoire* for remembering the formula
 - Understand a number of properties about the cross product



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The cross product

References

- [1] https://en.wikipedia.org/wiki/Cross_product
- [2] <https://en.wikipedia.org/wiki/Aide-m%C3%A9moire>

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The cross product

Acknowledgments

None so far.

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The cross product

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see <https://www.rbg.ca/> for more information.



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The cross product

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